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Exact soliton solutions to the classical non-Abelian Yang–Mills gauge field

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Abstract. We have derived a set of differential equations describing the spatial and temporary variation of the Yang–Mills gauge field within the O(4) group under the ansatz that the spin connection in R^4 space is simply our gauge potential and the line element of R^4 space is expressible in a diagonalizable form. We have also obtained four sets of analytical, special solutions to the potential and field strength components. These solutions demonstrate soliton-like properties. Though the form of the soliton is more complicated than that of the usual pulse-shaped one, these solitons are found to propagate with constant velocity c. The energy momentum tensor T_{44} is also analysed numerically; the form of T_{44} is found to be non-dispersive while the soliton propagates. Moreover, the total energy is found always to be positive definite.

1. Introduction

In the first paper of our series [1], we attempted to find classical solutions of the Yang–Mills (YM) gauge field equation in Euclidean spacetime within the differential geometry regime. In order to look for a special solution, we assigned a set of ansatz to the line element of R^4 space under the spherically symmetric condition. After a series of manipulations, the system of field equations were simplified to only a differential gauge field equation describing the variation of an O(4) non-Abelian gauge field.

Though we were not able to obtain analytical solution to the stated field equation, we have carried out a series of numerical simulations of the field equation $\ddot{f}+\dot{f}+f(1-f^2)=0$ with $\dot{f}\equiv \mathrm{d}f/\mathrm{d}z$, $r/r_0\equiv \mathrm{e}^{-z}$.

Thus, the field equation has a form analogous to that describing the motion of a classical particle in the presence of a damping force and a potential. Under certain boundary conditions we have discovered that the motion of the particle is confined in an oscillating manner within a finite spatial region. Since the gauge field component is $F^{\hat{r}\hat{\theta}r\theta}=(1/r^2)f'$, we have thus discovered that there is an inherent confinement property in the gauge field.

We also emphasize that, in the final part of our investigation, the line elements are not time-dependent and our solutions are static in the usual four-dimensional space. It is a natural procedure to extend our previous general differential geometrical method to the time-dependent problem relating to an O(4) group gauge potential. As the YM gauge field equation system is nonlinear, soliton solutions may exist. Previously, pseudoparticles like instantons, magnetic monopoles and melons have been related to soliton solutions. However, these solutions do not possess propagating properties

[6, 10]. In recent years, employing the inverse-scattering method, certain sets of soliton solutions of self-dual SU(2) and SU(3) gauge fields were found [11–13]. However, these solutions are very complex and it is very difficult to analyse the propagating characteristics and the energy variation in the usual manner practised by physicists.

In this investigation, we start with the line element in R^4 space and arrive at a system of O(4) gauge field equations. Adopting an ansatz relating the field potential and spin connection of R^4 space similar to that in [14–16], we are able to obtain four sets of analytical solutions to the YM gauge field. From our numerical evaluation of these solutions, we find that the solitons discovered are, in general, pulse-shaped ones with rather complex structures or are 'composite solitons', like that reported earlier for the study of gravitation solitons in the presence of a spherically symmetrical terrestrial field [17].

2. Derivation of the YM gauge field equations

In order to solve the gauge field equations under certain specified conditions and to analyse the properties of this gauge field, we introduce an O(4) gauge potential $A_\mu^{\hat\alpha\hat\beta}$, in the manner described in [10], where $\mu = 1$–4 are the usual spacetime indices and $\hat\alpha, \hat\beta = 1$–$4$ are the Lorentz indices. Note that $A_\mu^{\hat\alpha\hat\beta}$ is antisymmetric on $\hat\alpha, \hat\beta$. The O(4) group can be resolved into two SU(2) groups, $O(4) \sim SU(2) \times SU(2)$, and the SU(2) gauge potentials $A_\mu^{\hat i}$ ($i = 1$–3) related to the O(4) gauge potential are

$$\pm A_\mu^{\hat\alpha\hat\beta} = \tfrac{1}{2}(A_\mu^{\hat4\hat i} \pm \tfrac{1}{2}\varepsilon_{\hat i\hat j\hat k} A_\mu^{\hat j\hat k}). \tag{2.1}$$

Based on the O(4) gauge potential, we can obtain the O(4) gauge field strength tensor

$$F_{\mu\nu}^{\hat\alpha\hat\beta} = \partial_\mu A_\nu^{\hat\alpha\hat\beta} - \partial_\nu A_\mu^{\hat\alpha\hat\beta} + A_\mu^{\hat\alpha\hat\gamma} A_\nu^{\hat\gamma\hat\beta} - A_\nu^{\hat\alpha\hat\gamma} A_\mu^{\hat\gamma\hat\beta}. \tag{2.2}$$

On the other hand, the gauge field equation in Cartesian coordinates and in flat spacetime can be expressed as

$$\partial_\nu F^{\hat\alpha\hat\beta\mu\nu} + A_\nu^{\hat\alpha\hat\gamma} F^{\hat\gamma\hat\beta\mu\nu} + A_\nu^{\hat\beta\hat\gamma} F^{\hat\alpha\hat\gamma\mu\nu} = 0. \tag{2.3}$$

In order to integrate the gauge field equation (2.3), one usually takes an ansatz to simplify it. For example, the 't Hooft ansatz is [6]

$$A_\mu = \sigma_{\mu\nu} \frac{\phi_\nu}{\phi} \tag{2.4}$$

which leads to the instanton solution. Thus, it is fruitful to find suitable ansatze, arriving at certain solutions as accomplished in the past [2–9]. However, at present, workable ansatze arise from intuition, rather than standard methodology.

In this investigation, we propose a new ansatz based on a logical idea in the framework of differential geometry. First, we construct in R^4 space a spin connection whose geometrical properties are equivalent to that of the gauge potential in the O(4) group. We can therefore assign this spin connection to our O(4) gauge potential. The first concrete process along this line is to express the line element $\mathrm{d}s$ in R^4 space as

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \tag{2.5}$$

where $\mu, \nu = 1$–4. Within R^4 space, we introduce the Euclidean metric

$$\mathrm{d}s^2 = \delta_{\hat\alpha\hat\beta}\, \mathrm{d}y^{\hat\alpha}\, \mathrm{d}y^{\hat\beta} \tag{2.6}$$

in which $\mathrm{d}y^{\hat\alpha}$ is a unit orthogonal basis, with $\hat\alpha, \hat\beta = 1$–$4$.

As pointed out by Wilczek [16], we can understand certain crucial properties of the gauge field by the introduction of the Vierbein field $L_\mu^{\hat\alpha}$, which is an orthogonal transformation relating the unit orthogonal basis $\mathrm{d}y^{\hat\alpha}$ and another basis $\mathrm{d}x^\mu$:

$$\mathrm{d}y^{\hat\alpha} = L_\mu^{\hat\alpha}\,\mathrm{d}\mu. \tag{2.7}$$

$L_\mu^{\hat\alpha}$ satisfies the orthogonal condition

$$L_\mu^{\hat\alpha} L^{\mu\hat\beta} = \delta_{\hat\alpha\hat\beta}. \tag{2.8}$$

In view of (2.5)-(2.7), the metric $g_{\mu\nu}$ and the Vierbein field are related by

$$g_{\mu\nu} = L_\mu^{\hat\alpha} L_\nu^{\hat\beta}\delta_{\hat\alpha\hat\beta}. \tag{2.9}$$

From the metric $g_{\mu\nu}$, we can write down the Riemann connection in R^4:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}). \tag{2.10}$$

Carrying out an orthogonal frame transformation on the Riemann connection $\Gamma^\sigma_{\mu\nu}$ by the Vierbein field $L_\mu^{\hat\alpha}$, we derive the explicit form of our spin connection:

$$C_\mu^{\hat\alpha\hat\beta} = \tfrac{1}{2}L^{\lambda\hat\beta}(L^{\hat\alpha}_{\mu,\lambda} - L^{\hat\alpha}_{\lambda,\mu}) + \tfrac{1}{2}L^{\sigma\hat\alpha}(L^{\hat\beta}_{\sigma,\mu} - L^{\hat\beta}_{\mu,\sigma}) + \tfrac{1}{2}L^{\sigma\hat\alpha}L^{\lambda\hat\beta}(L^{\hat\varepsilon}_{\sigma,\lambda}L^{\hat\varepsilon}_{\mu} - L^{\hat\varepsilon}_{\lambda,\sigma}L^{\hat\varepsilon}_{\mu}). \tag{2.11}$$

It can be proved that [15, 16] the spin connection $C^{\hat\alpha\hat\beta}$ is invariant under simultaneous rotations of the frames x^μ and $y^{\hat\alpha}$. On the other hand, the O(4) gauge potential also has symmetrical synchronization properties; namely, in physical systems, $A_\mu^{\hat\alpha\hat\beta}$ is invariant under simultnaeous rotations of spacetime and isotopic space. We can therefore assign

$$A_\mu^{\hat\alpha\hat\beta} = C_\mu^{\hat\alpha\hat\beta}. \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.2) and (2.3) we can obtain the field strength and the gauge field equation, and eventually arrive at our analytical solutions.

3. Analytical soliton solutions to the O(4) gauge field equations

We limit ourselves to the diagonalized form of the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} B^2(u) & & & \\ & C^2(u) & & \\ & & D^2(u) & \\ & & & A^2(u) \end{pmatrix} \tag{3.1}$$

where $u \equiv ct - z$. Since $g_{\mu\nu}$ is only a function of u, we shall study the solutions corresponding to the gauge field which propagate along the z-direction with velocity c.

First, from (2.5), we can write the form of the Vierbein field:

$$L_\mu^{\hat\alpha} = \begin{pmatrix} B(u) & & & \\ & C(u) & & \\ & & D(u) & \\ & & & A(u) \end{pmatrix}$$

$$L^{\mu\hat\alpha} = \begin{pmatrix} B^{-1}(u) & & & \\ & C^{-1}(u) & & \\ & & D^{-1}(u) & \\ & & & A^{-1}(u) \end{pmatrix}. \tag{3.2}$$

Substituting (3.2) into (2.12), we arrive at the relevant gauge potential whose non-zero components are

$$\begin{aligned} A_1^{\hat{1}\hat{3}} &= -\frac{B'}{D} & A_1^{\hat{1}\hat{4}} &= -\mathrm{i}\,\frac{B'}{A} \\ A_2^{\hat{2}\hat{3}} &= -\frac{C'}{D} & A_2^{\hat{2}\hat{4}} &= -\mathrm{i}\,\frac{C'}{A} \\ A_3^{\hat{3}\hat{4}} &= -\mathrm{i}\,\frac{D'}{A} & A_4^{\hat{3}\hat{4}} &= \frac{A'}{D} \end{aligned} \tag{3.3}$$

where the prime indicates differentiation with respect to u. It is tedious, but elementary, to obtain the explicit expressions for the non-zero components of the gauge field strength:

$$\begin{aligned} F_{12}^{\hat{1}\hat{2}} &= B'C'\left(\frac{1}{A^2}-\frac{1}{D^2}\right) \\ F_{13}^{\hat{1}\hat{3}} &= -\left[\left(\frac{B'}{D}\right)'-\frac{B'D'}{A^2}\right] \\ F_{13}^{\hat{1}\hat{4}} &= -\mathrm{i}\left[\left(\frac{B'}{A}\right)'-\frac{B'D'}{AD}\right] \\ F_{14}^{\hat{1}\hat{3}} &= +\mathrm{i}\left[\left(\frac{B'}{D}\right)'-\frac{A'B'}{AD}\right] \\ F_{14}^{\hat{1}\hat{4}} &= \left(\frac{B'}{A}\right)'-\frac{A'B'}{D^2} \\ F_{23}^{\hat{2}\hat{3}} &= -\left[\left(\frac{C'}{D}\right)'-\frac{C'D'}{A^2}\right] \\ F_{24}^{\hat{2}\hat{3}} &= -\mathrm{i}\left[\left(\frac{C'}{D}\right)'-\frac{A'C'}{AD}\right] \\ F_{23}^{\hat{2}\hat{4}} &= -\mathrm{i}\left[\left(\frac{C'}{A}\right)'-\frac{C'D'}{AD}\right] \\ F_{24}^{\hat{2}\hat{4}} &= \left(\frac{C'}{A}\right)'-\frac{A'C'}{D^2} \\ F_{34}^{\hat{3}\hat{4}} &= -\left[\left(\frac{A'}{D}\right)'-\left(\frac{D'}{A}\right)'\right]. \end{aligned} \tag{3.4}$$

In our formalism, substituting (3.3) and (3.4) into (2.3), we derive our set of gauge field equations:

$$\left[\frac{B'}{A}\left(\frac{A'}{D}-\frac{D'}{A}\right)\right]'+\frac{B'C'^2}{A}\left(\frac{1}{A^2}-\frac{1}{D^2}\right)+\left(\frac{B'}{A}\right)'\left(\frac{A'}{D}-\frac{D'}{A}\right)+\frac{B'}{D}\left(\frac{D'^2}{A^2}-\frac{A'^2}{D^2}\right)=0 \tag{3.5}$$

$$\left[\frac{B'}{D}\left(\frac{D'}{A}-\frac{A'}{D}\right)\right]'+\frac{B'C'^2}{A}\left(\frac{1}{A^2}-\frac{1}{D^2}\right)+\left(\frac{B'}{D}\right)'\left(\frac{A'}{D}-\frac{D'}{A}\right)+\frac{B'}{A}\left(\frac{D'^2}{A^2}-\frac{A'^2}{D^2}\right)=0 \tag{3.6}$$

$$\left[\frac{C'}{A}\left(\frac{A'}{D}-\frac{D'}{A}\right)\right]'+\frac{B'^2C'}{D}\left(\frac{1}{A^2}-\frac{1}{D^2}\right)+\left(\frac{C'}{A}\right)'\left(\frac{A'}{D}-\frac{D'}{A}\right)+\frac{C'}{D}\left(\frac{D'^2}{A^2}-\frac{A'^2}{D^2}\right)=0 \tag{3.7}$$

$$\left[\frac{C'}{D}\left(\frac{A'}{D}-\frac{D'}{A}\right)\right]'-\frac{B'^2C'}{A}\left(\frac{1}{A^2}-\frac{1}{D^2}\right)+\left(\frac{C'}{D}\right)'\left(\frac{A'}{D}-\frac{D'}{A}\right)+\frac{C'}{A}\left(\frac{D'^2}{A^2}-\frac{A'^2}{D^2}\right)=0 \tag{3.8}$$

$$\left(\frac{A'}{D}-\frac{D'}{A}\right)''+\frac{B'^2}{A^2}\left(\frac{D'}{A}-\frac{A'}{D}\right)+\frac{C'^2}{A^2}\left(\frac{D'}{A}-\frac{A'}{D}\right)=0 \tag{3.9}$$

$$\left(\frac{A'}{D}-\frac{D'}{A}\right)''-\frac{B'^2}{D^2}\left(\frac{A'}{D}-\frac{D'}{A}\right)-\frac{C'^2}{D^2}\left(\frac{A'}{D}-\frac{D'}{A}\right)=0. \tag{3.10}$$

In (3.5)–(3.10) there are four unknowns but six equations, implying that the solutions might be overdetermined. In such a situation we first look for restriction(s) for which special solutions can be obtained. In particular, we observe that if

$$\frac{A'}{D}-\frac{D'}{A}=0 \tag{3.11}$$

the equations in (3.5)–(3.10) are greatly simplified and we arrive at the following two special solutions:

$$\begin{aligned}&D^2=A^2+a\\&C=k\\&B \text{ is an arbitrary function of } u\end{aligned} \tag{3.12a}$$

where a, k are constants, and

$$\begin{aligned}&D=\pm A\\&B, C \text{ are arbitrary functions of } u.\end{aligned} \tag{3.12b}$$

In general, there are an infinite number of solutions of A, B, C, D which satisfy (3.12a) or (3.12b). However, inspection of (3.3) and (3.4) shows that, there, A and D appear as denominators of a number of terms. There is the physical requirement that each component of the gauge field potential and field strength in (3.3) and (3.4) must tend to zero, as $u=ct+z\to\pm\infty$. Along this line we must note that, in assigning the boundary condition to $\pm\infty$, we refer to the situation where time is finite. In other words, $u\to\pm\infty$ pertains to a spatial boundary condition; this is the usual practice in physical systems and such a concept was adopted when the instanton solution was obtained. Under such a constraint, the functions A, B, C, D should satisfy the following conditions: (i) The zero points of A, D are excluded in our solutions; (ii) A, B, C, D are all continuous and differentiable within the whole defined range of u; (iii) B'/D, B'/A, C'/D, C'/A, D'/A, A'/D all tend to zero as $u\to\pm\infty$. Moreover, based on the explicit forms of the potential and field strength components, as in (3.3) and (3.4), the singular points specified by $A=0$ or $D=0$ must be excluded.

We note that if A, B, C, D are simply trigonometric functions, conditions (i), (ii), (iii) cannot be satisfied simultaneously. In other words, A, B, C, D cannot be expressed as combinations of monochromatic plane wave solutions. We observe that even before arriving at our analytical solutions, the field strength components exist only in a confined region of μ.

Based on the above boundary conditions, we have chosen four sets of solutions for the functions A, B, C, D. The corresponding non-zero solutions to the potential and field strength components can be obtained from (3.3) and (3.4). These four sets, (i)-(iv), are as follows:

(i)
$$\begin{cases} A(u)=D(u)=u^4-u^2+1 \\ B(u)=\operatorname{sech}(u) \\ C(u)=\tanh(u). \end{cases}$$

$$\begin{cases} A_1^{\hat{1}\hat{3}}=\dfrac{\operatorname{sech}(u)\tanh(u)}{u^4-u^2+1}=-\mathrm{i}A_1^{\hat{1}\hat{4}} \\ A_2^{\hat{2}\hat{3}}=-\dfrac{\operatorname{sech}^2(u)}{u^4-u^2+1}=-\mathrm{i}A_2^{\hat{2}\hat{4}} \\ A_4^{\hat{3}\hat{4}}=\dfrac{4u^3-2u}{u^4-u^2+1}=\mathrm{i}A_3^{\hat{3}\hat{4}}. \end{cases}$$

$$\begin{cases} F_{13}^{\hat{1}\hat{3}}=-\dfrac{2(4u^3-2u)\operatorname{sech}(u)\tanh(u)}{(u^4-u^2+1)^2}-\dfrac{\operatorname{sech}(u)-2\operatorname{sech}^3(u)}{u^4-u^2+1} \\ \qquad =-F_{14}^{\hat{1}\hat{4}}=-\mathrm{i}F_{14}^{\hat{1}\hat{3}}=-\mathrm{i}F_{13}^{\hat{1}\hat{4}} \\ F_{23}^{\hat{2}\hat{3}}=\dfrac{2(4u^3-2u)\operatorname{sech}^2(u)}{(u^4-u^2+1)^2}+\dfrac{2\operatorname{sech}^2(u)\tanh(u)}{u^4-u^2+1} \\ \qquad =-F_{24}^{\hat{2}\hat{4}}=-\mathrm{i}F_{24}^{\hat{2}\hat{3}}=-\mathrm{i}F_{23}^{\hat{2}\hat{4}}. \end{cases}$$

(ii)
$$\begin{cases} A(u)=-\dfrac{(u^4+\frac{1}{2})^{1/2}}{u^2+1} \\ D(u)=\dfrac{(2u^4+2u^2+\frac{3}{2})^{1/2}}{u^2+1} \\ B(u)=\operatorname{sech}(u)\tanh(u) \\ C(u)=1. \end{cases}$$

$$\begin{cases} A_1^{\hat{1}\hat{3}}=-\dfrac{2\operatorname{sech}^3(u)-\operatorname{sech}(u)}{(2u^4+2u^2+\frac{3}{2})^{1/2}}(u^2+1) \\ A_1^{\hat{1}\hat{4}}=\mathrm{i}\dfrac{2\operatorname{sech}^3(u)-\operatorname{sech}(u)}{(u^4+\frac{1}{2})^{1/2}}(u^2+1) \\ A_4^{\hat{3}\hat{4}}=\dfrac{-2u(u^2-\frac{1}{2})}{(u^2+1)[(u^4+\frac{1}{2})(2u^4+2u^2+\frac{3}{2})]^{1/2}}=\mathrm{i}A_3^{\hat{3}\hat{4}}. \end{cases}$$

$$
\begin{cases}
F_{13}^{\hat1\hat3} = \dfrac{\operatorname{sech}(u)}{(2u^4+2u^2+\frac{3}{2})^{1/2}}\Big[\tanh(u)(u^2+1)[6\operatorname{sech}^2(u)-1] \\
\qquad\qquad + [2\operatorname{sech}^2(u)-1](2u^3-3u)\left(\dfrac{1}{u^4+\frac{1}{2}}+\dfrac{1}{2u^4+2u^2+\frac{3}{2}}\right)\Big] \\
\qquad = -\mathrm{i}F_{13}^{\hat1\hat4} \\
F_{14}^{\hat1\hat4} = \dfrac{\operatorname{sech}(u)}{(u^4+\frac{1}{2})^{1/2}}\Big[\tanh(u)(u^2+1)[6\operatorname{sech}^2(u)-1] \\
\qquad\qquad + [2\operatorname{sech}^2(u)-1](2u^3-u)\left(\dfrac{1}{u^4+\frac{1}{2}}+\dfrac{1}{2u^4+2u^2+\frac{3}{2}}\right)\Big] \\
\qquad = \mathrm{i}F_{13}^{\hat1\hat4}.
\end{cases}
$$

(iii)
$$
\begin{cases}
A(u) = \tanh^2(u)\operatorname{sech}(u)+1 \\
D(u) = \{[\tanh^2(u)\operatorname{sech}(u)+1]^2+1\}^{1/2} \\
B(u) = \operatorname{sech}(u) \\
C(u) = 1.
\end{cases}
$$

$$
\begin{cases}
A_1^{\hat1\hat3} = \dfrac{\operatorname{sech}(u)\tanh(u)}{\{[\tanh^2(u)\operatorname{sech}(u)+1]^2+1\}^{1/2}} \\
A_1^{\hat1\hat4} = \mathrm{i}\dfrac{\operatorname{sech}(u)\tanh(u)}{\tanh^2(u)\operatorname{sech}(u)+1} \\
A_4^{\hat3\hat4} = \dfrac{\tanh^3(u)\operatorname{sech}^2(u)[2-3\tanh^2(u)]}{\{[\tanh^2(u)\operatorname{sech}(u)+1]^2+1\}^{1/2}} = \mathrm{i}A_3^{\hat3\hat4}.
\end{cases}
$$

$$
\begin{cases}
F_{13}^{\hat1\hat3} = -\dfrac{\operatorname{sech}(u)}{\{1+[\tanh^2(u)\operatorname{sech}(u)]^2\}^{1/2}}\Big[1-2\operatorname{sech}^2(u) \\
\qquad\qquad + \tanh^2(u)\operatorname{sech}(u)[2-3\tanh^2(u)][1+\tanh^2(u)\operatorname{sech}(u)] \\
\qquad\qquad \times\left(\dfrac{1}{[1+\tanh^2(u)\operatorname{sech}(u)]^2}+\dfrac{1}{1+[1+\tanh^2(u)\operatorname{sech}(u)]^2}\right)\Big] \\
\qquad = -\mathrm{i}F_{14}^{\hat1\hat3} \\
F_{14}^{\hat1\hat4} = \operatorname{sech}(u)\Big[\dfrac{1-2\operatorname{sech}^2(u)}{1+\tanh^2(u)\operatorname{sech}(u)}+\tanh^2(u)\operatorname{sech}(u)[2-3\tanh^2(u)] \\
\qquad\qquad \times\left(\dfrac{1}{[1+\tanh^2(u)\operatorname{sech}(u)]^2}+\dfrac{1}{1+[1+\tanh^2(u)\operatorname{sech}(u)]^2}\right)\Big] \\
\qquad = -\mathrm{i}F_{13}^{\hat1\hat4}.
\end{cases}
$$

(iv)
$$\begin{cases} A(u)=\tanh(u)+2 \\ D(u)=\{[\tanh(u)+2]^2+1\}^{1/2} \\ B(u)=\tanh^2(u)\,\mathrm{sech}(u) \\ C(u)=1. \end{cases}$$

$$\begin{cases} A_1^{\hat{1}\hat{3}}=-\dfrac{\tanh(u)\,\mathrm{sech}(u)[2\,\mathrm{sech}^2(u)-\tanh^2(u)]}{\{[\tanh(u)+2]^2+1\}^{1/2}} \\ A_1^{\hat{1}\hat{4}}=-\mathrm{i}\dfrac{\tanh(u)\,\mathrm{sech}(u)[2\,\mathrm{sech}^2(u)-\tanh^2(u)]}{\tanh(u)+2} \\ A_4^{\hat{3}\hat{4}}=\dfrac{\mathrm{sech}^2(u)}{\{[\tanh(u)+2]^2+1\}^{1/2}}=\mathrm{i}A_3^{\hat{3}\hat{4}}. \end{cases}$$

$$\begin{cases} F_{13}^{\hat{1}\hat{3}}=\dfrac{-\mathrm{sech}(u)}{\{1+[2+\tanh(u)]^2\}^{1/2}}\Bigg[\tanh^4(u)+\mathrm{sech}^2(u)[11\,\mathrm{sech}^2(u)-9] \\ \qquad +[2-3\tanh^2(u)][2+\tanh(u)]\tanh(u)\,\mathrm{sech}^2(u) \\ \qquad \times\left(\dfrac{1}{[2+\tanh(u)]^2}+\dfrac{1}{1+[2+\tanh(u)]^2}\right)\Bigg] \\ \qquad =-\mathrm{i}F_{14}^{\hat{1}\hat{3}} \\ F_{14}^{\hat{1}\hat{4}}=\mathrm{sech}(u)\Bigg[\dfrac{\tanh^4(u)+\mathrm{sech}^2(u)[11\,\mathrm{sech}^2(u)-9]}{2+\tanh(u)} \\ \qquad -\tanh(u)\,\mathrm{sech}^2(u)[2-3\tanh^2(u)] \\ \qquad \times\left(\dfrac{1}{[2+\tanh(u)]^2}+\dfrac{1}{1+[2+\tanh(u)]^2}\right)\Bigg] \\ \qquad =\mathrm{i}F_{13}^{\hat{1}\hat{4}}. \end{cases}$$

4. Analysis of solutions

In figures 1(a) and 1(b), we plot the non-zero $A_\mu^{\hat{\alpha}\hat{\beta}}$ versus z and $F_{\mu\sigma}^{\hat{\alpha}\hat{\beta}}$ versus z components for set (i) solutions at $t=0$. Clearly, some of the components are equal and there are only three different potential components and two different field strength components. As the velocity of propagation is c, these 'composite pulse-shaped' solitons travel to the right at $t>0$. Similarly, the non-zero $A_\mu^{\hat{\alpha}\hat{\beta}}$ versus z and $F_{\mu\sigma}^{\hat{\alpha}\hat{\beta}}$ versus z solutions are indicated for sets (ii)–(iv) in figures 2–4, respectively. Many soliton solutions in well known nonlinear equations (e.g. KdV, modified KdV) are single pulse or single kink–antikink types of solutions. Here we have obtained solutions with more complex features, as shown in figures 1–4. We cannot analyse the structure of such composite solitons and deduce meaningful physical characteristics until we investigate the analogy between our solitons and physical particles.

We note that some of the potential and field strength components are imaginary members. This result is quite obvious as, in Euclidean space, the time variable is

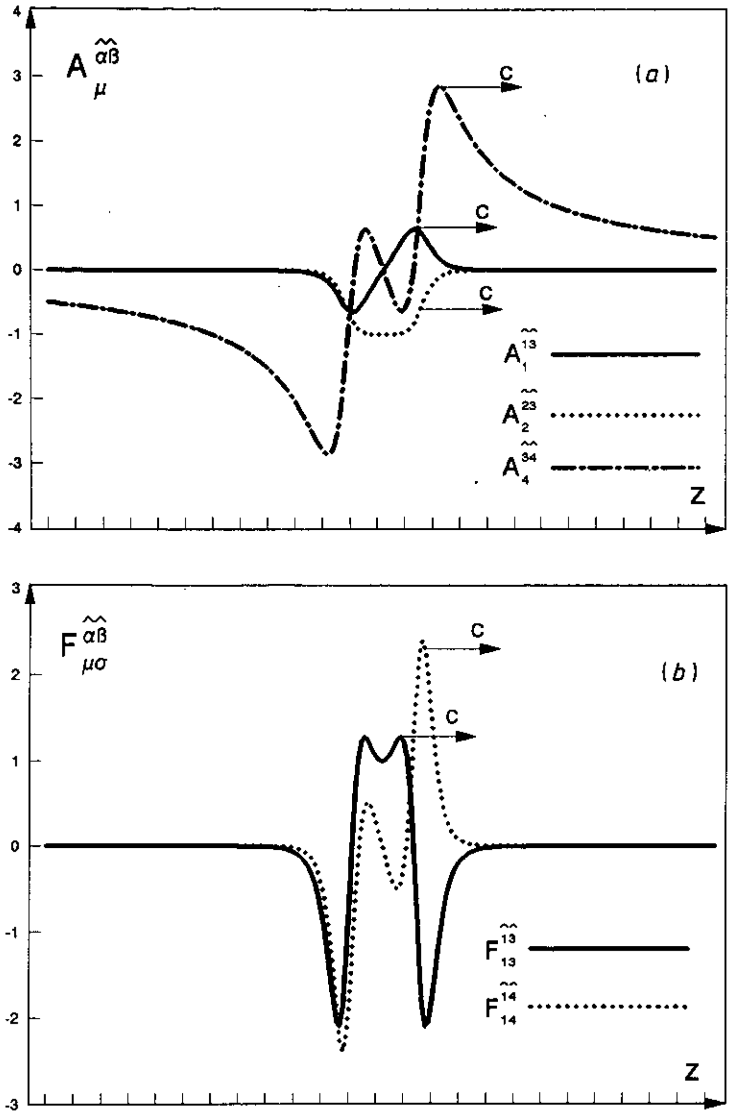


Figure 1. (a) Potential components $A_1^{\hat{1}\hat{3}}$, $A_2^{\hat{2}\hat{3}}$, $A_4^{\hat{3}4}$ versus the spatial variable z for solution set (i). (b) Field strength components $F_{13}^{\hat{1}\hat{3}}$, $F_{23}^{\hat{2}\hat{3}}$ versus z for solution set (i).

imaginary while the propagation factor u is real; the appearance of the imaginary number i in $A_\mu^{\hat{\alpha}\hat{\beta}}$, $F_{\mu\nu}^{\hat{\alpha}\hat{\beta}}$ is natural. If we express our system in Minkowsky space, we expect i to be eliminated.

5. Distribution of energy density in our O(4) YM gauge field

Starting from the Lagrangian in the gauge field

$$L = -\tfrac{1}{4} F_{\mu\nu}^{\hat{\alpha}\hat{\beta}} F_{\mu\nu}^{\hat{\alpha}\hat{\beta}} \tag{4.1}$$

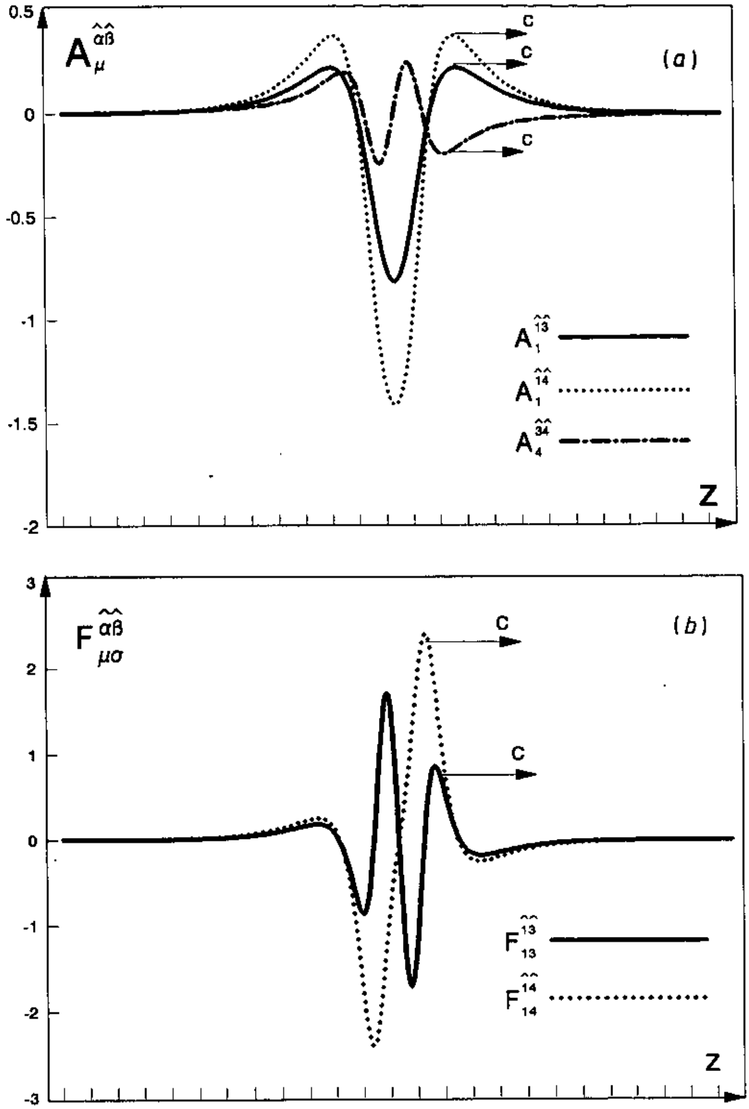


Figure 2. (*a*) Potential components $A_1^{\hat{1}\hat{3}}$, $A_1^{\hat{1}\hat{4}}$, $A_4^{\hat{3}\hat{4}}$ versus the spatial variable z for solution set (ii). (*b*) Field strength components $F_{14}^{\hat{1}\hat{3}}$, $f_{14}^{\hat{1}\hat{4}}$ versus z for solution set (ii).

we can write down the energy–momentum tensor of the field:

$$T_{\mu\nu} = F^{\hat{\alpha}\hat{\beta}}_{\mu\sigma} F^{\hat{\alpha}\hat{\beta}}_{\nu\sigma} + \tfrac{1}{4}\delta_{\mu\nu} F^{\hat{\alpha}\hat{\beta}}_{\tau\sigma} F^{\tau\sigma}_{\hat{\alpha}\hat{\beta}}. \tag{4.2}$$

The tensor $T_{\mu\nu}$ must satisfy the following conservation law:

$$\partial_\mu T_{\mu\nu} = 0. \tag{4.3}$$

Drawing an analogy between the electromagnetic field and the YM gauge field, we can write the non-Abelian electric field tensor as

$$E_i^{\hat{\alpha}\hat{\beta}} = F_{4i}^{\hat{\alpha}\hat{\beta}} \tag{4.4a}$$

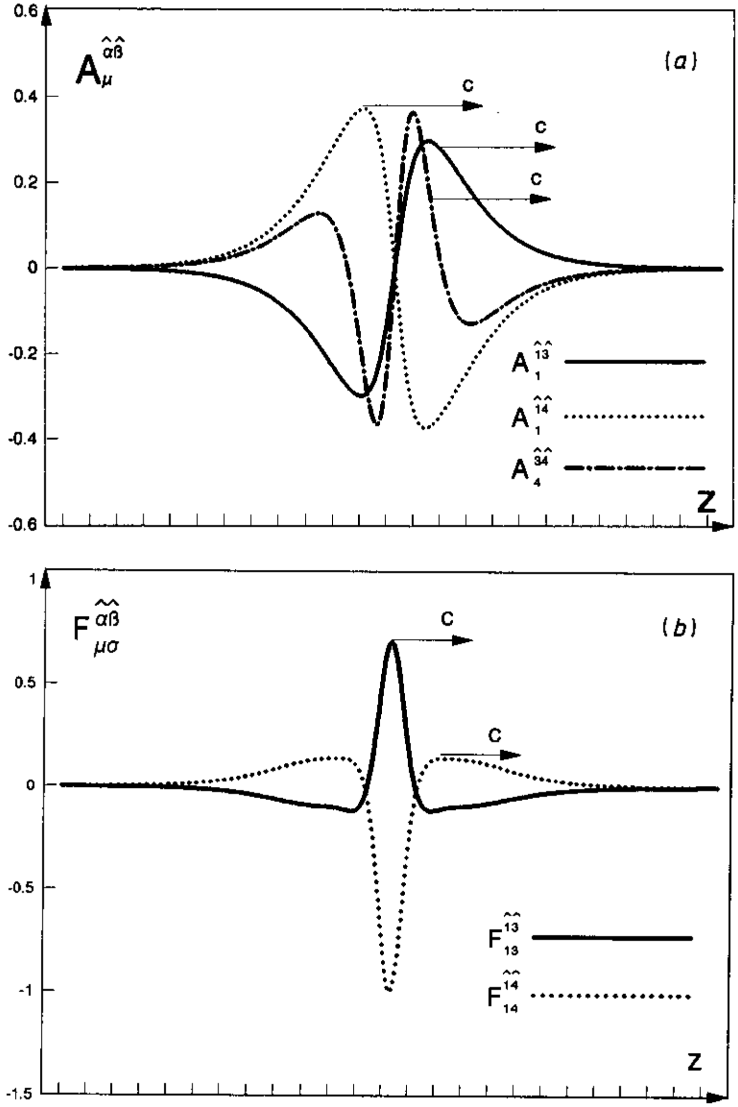


Figure 3. (*a*) Potential components $A_1^{\hat{1}\hat{3}}$, $A_1^{\hat{1}\hat{4}}$, $A_4^{\hat{3}\hat{4}}$ versus the spatial variable z for solution set (iii). (*b*) Field strength components $F_{13}^{\hat{1}\hat{3}}$, $F_{14}^{\hat{1}\hat{4}}$ versus z for solution set (iii).

and the magnetic field tensor as

$$B_i^{\hat{\alpha}\hat{\beta}} = -\tfrac{1}{2}\varepsilon_{ijk}F_{jk}^{\hat{\alpha}\hat{\beta}}. \tag{4.4b}$$

Substituting (4.4) into (4.2), we obtain the corresponding energy density component

$$T_{44} = \tfrac{1}{2}(E_i^{\hat{\alpha}\hat{\beta}}E_i^{\hat{\alpha}\hat{\beta}} + B_i^{\hat{\alpha}\hat{\beta}}B_i^{\hat{\alpha}\hat{\beta}}). \tag{4.5}$$

Clearly, we can demonstrate the T_{44} versus z relation using our four sets of particular solutions and relations (4.4) and (4.5).

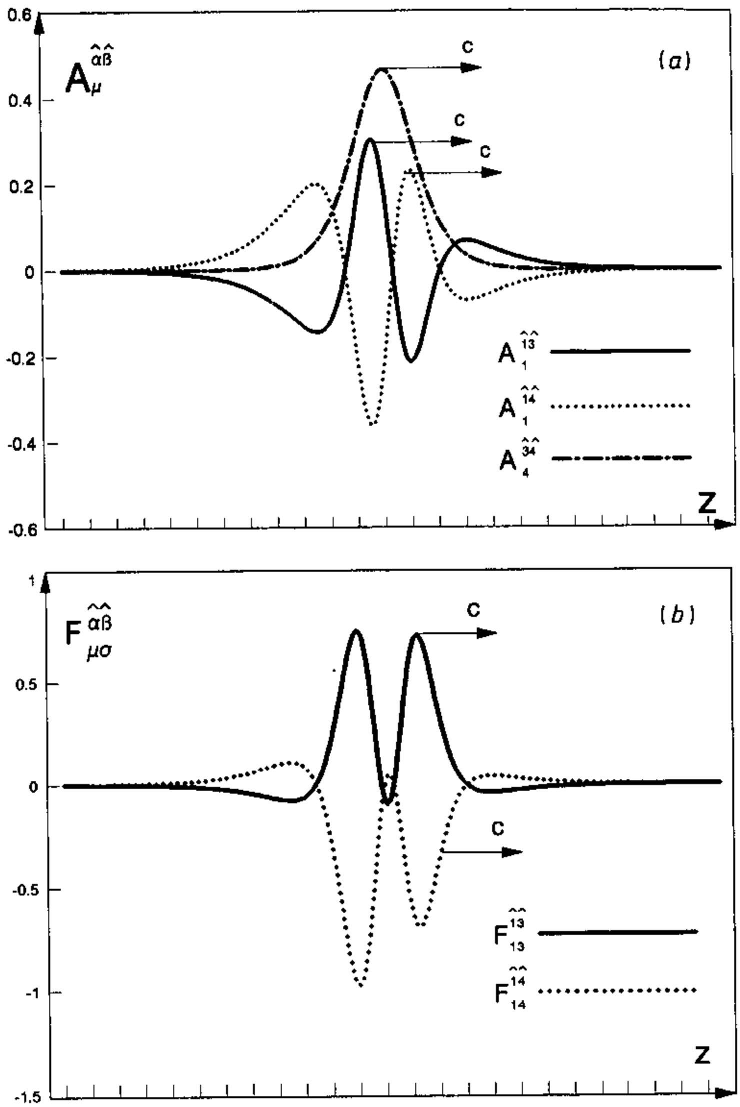
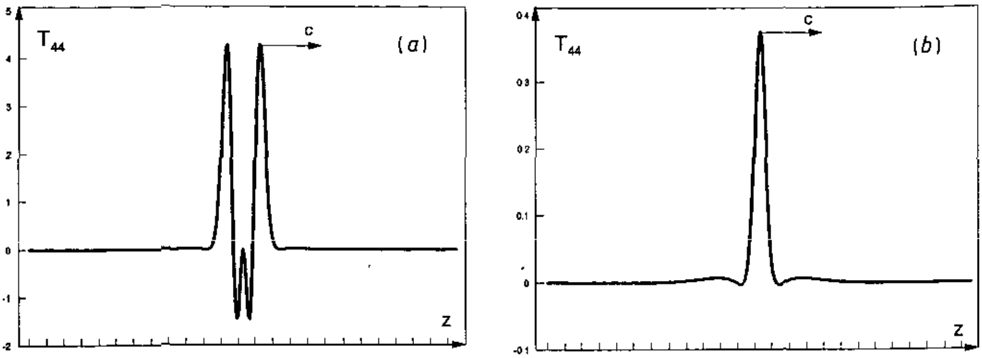


Figure 4. (*a*) Potential components $A_1^{\hat1\hat3}$, $A_1^{\hat1\hat4}$, $A_4^{\hat3\hat4}$ versus the spatial variable z for solution set (iv). (*b*) Field strength components $F_{13}^{\hat1\hat3}$, $F_{14}^{\hat1\hat4}$ versus z for solution set (iv).

It is interesting to remark that for set (i) solutions, $F_{13}^{\hat1\hat3}=-F_{14}^{\hat1\hat4}=-\mathrm{i}F_{14}^{\hat1\hat3}=-\mathrm{i}F_{13}^{\hat1\hat4}$ and $F_{23}^{\hat2\hat3}=-F_{24}^{\hat2\hat4}=-\mathrm{i}F_{24}^{\hat2\hat3}=-\mathrm{i}F_{23}^{\hat2\hat4}$. In view of (4.4*a*) and (4.4*b*), we find that $F_{13}^{\hat1\hat3}=-B_2^{\hat1\hat3}$, $F_{14}^{\hat1\hat4}=-E_1^{\hat1\hat4}$, $F_{14}^{\hat1\hat3}=-E_1^{\hat1\hat3}$, $F_{13}^{\hat1\hat4}=-B_2^{\hat1\hat4}$, $F_{23}^{\hat2\hat3}=B_1^{\hat2\hat3}$, $F_{24}^{\hat2\hat4}=-E_2^{\hat2\hat4}$, $F_{24}^{\hat2\hat3}=-E_2^{\hat2\hat3}$ and $F_{23}^{\hat2\hat4}=B_1^{\hat2\hat4}$. It is easy to show that while the electric and magnetic fields are non-zero in general, the electric energy in Euclidean space is zero when we add the energies corresponding to the various components together (due to the existence of i), indicating that the energy density is zero; this property also occurs for the magnetic energy sum. In other words, (4.5) gives, in our case, $T_{44}=\frac{1}{2}(0+0)=0$. In the self-dual situation of the previous instanton study [14], $E_i^{\hat\alpha\hat\beta}E_i^{\hat\alpha\hat\beta}=-B_i^{\hat\alpha\hat\beta}B_i^{\hat\alpha\hat\beta}$, leading to $T_{44}=0$. Thus, the energy density is zero for both cases, but for different reasons in Euclidean space.



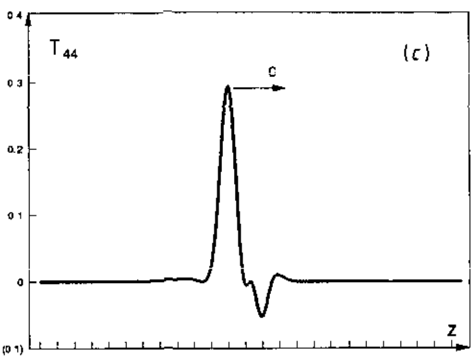


Figure 5. Momentum-energy pseudo-tensor components T_{44} versus z for: (a) solution set (ii); (b) solution set (iii); (c) solution set (iv).

The T_{44} versus z relations for our solutions sets (ii)–(iii) and (iv) are shown in figures 5(a), (b), (c) respectively. Here the total energy density is positive-definite in each case. We observe an energy pulse going along the z-direction with velocity c.

5. Conclusions

(i) In this investigation we have derived a set of differential equations relating the 'field functions' A, B, C, D, adopting an ansatz as specified in section 2. This set of six equations contain only four variable functions, and there exist certain overdetermined properties in our system. Based on this set of differential equations, we obtained the representation of the YM gauge field components and the field strength components, as presented in (3.3) and (3.4).

(ii) Needless to say, it is difficult to obtain general solutions to $A_\mu^{\hat{\alpha}\hat{\beta}}$ and $F_{\mu\nu}^{\hat{\alpha}\hat{\beta}}$ and we can only look for special solutions at this stage. One obvious choice is to confine ourselves to the situation where $A'/D - D'/A = 0$. We are then able to obtain analytical solutions to the potential and field strength components by fixing the forms of the arbitrary functions B and C. Since we are looking for soliton solutions, we choose B and C to take hypogeometrical forms. However, we must emphasize that the boundary conditions discussed in section 3 must be satisfied when choosing B and C. We thus believe that there exist certain inherent soliton-like property in our set of field equations.

(iii) We have found that both the potential and field strength components appear as 'composite pulse-shaped' solitons propagating with velocity c in the z-direction.

(iv) We have also written down the energy-momentum pseudo-tensor for the YM gauge field and have plotted the energy density versus z relation for our sets of solutions found. The first set of solution leads to a zero value of T_{44}, but based on reasons different from that in the instanton case of the self-dual gauge field. It would be fruitful to find out the meaning of $T_{44}=0$ in Minkowsky space for both these situations in the future.

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